On maximal connected I-spaces

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Maximal connected spaces

Definition

A topological space is called

- *maximal connected* [Thomas, 1968] if it is connected and has no connected strict expansion;
- *essentially connected* [Guthrie–Stone, 1973] if it is connected and every connected expansion has the same connected subsets.

Maximal connected spaces

Facts

- Both maximal connected spaces and essentially connected spaces are stable under connected subspaces. [Guthrie–Reynolds–Stone, 1973], [Guthrie–Stone, 1973]
- The real line is essentially connected [Hildebrand, 1967] and it has a maximal connected expansion [Simon, 1978], [Guthrie–Stone–Wage, 1978].
- No Hausdorff connected space with a dispersion point has a maximal connected expansion. [Guthrie–Stone, 1973]
- There are Hausdorff maximal connected spaces, but it is not known whether there are nondegenerate regular maximal connected spaces.

Implications between the classes

Definition

Recall the following properties of a topological space X.

- X is *submaximal* if every its dense subset is open.
- X is *nodec* if every its nowhere dense subset is closed.
- X is *irresolvable* if it has no two disjoint dense subsets.
- X is $T_{\frac{1}{2}}$ if every its singleton is open or closed.

We have the following implications.

$$\begin{tikzcd}
\text{maximal connected} \arrow[rr] \arrow[dr] & & \text{submaximal} \arrow[rr] \arrow[dr] \arrow[drrr] & & \text{nodec} & \\
& \text{essentially connected} & & T_{\frac{1}{2}} & & \text{hereditarily irresolvable}
\end{tikzcd}$$

Finitely generated spaces

Definition

A topological space X is called *finitely generated* or *Alexandrov* if every intersection of open sets is open. Equivalently, if

$$\overline{A} = \bigcup_{x \in A} \overline{\{x\}} \quad \text{for every } A \subseteq X.$$

Finitely generated maximal connected spaces

[Thomas, 1968] characterized finitely generated maximal connected spaces, we may reformulate the characterization as follows.

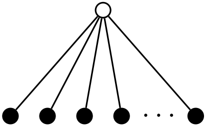
Proposition

Let X be a finitely generated $T_{\frac{1}{2}}$ space. Let $I(X)$ be the set of all isolated points.

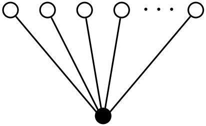
- The topology is uniquely determined by the bipartite graph G_X with bipartition $\langle I(X),\ X \setminus I(X)\rangle$ and with an edge between $x \in I(X)$ and $y \in X \setminus I(X)$ if and only if $\overline{\{x\}} \ni y$.
- X is connected $\iff$ G_X is connected as a graph.
- X is maximal connected $\iff$ G_X is a tree.

Therefore, finitely generated maximal connected spaces correspond to trees with fixed ordered bipartition.

Finitely generated maximal connected spaces

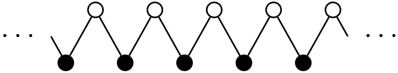


A principal ultrafilter space.



A principal ultraideal space.

The Sierpiński space.



The Khalimsky line.

Figure: Examples of finitely generated maximal connected spaces.

Finitely generated maximal connected spaces

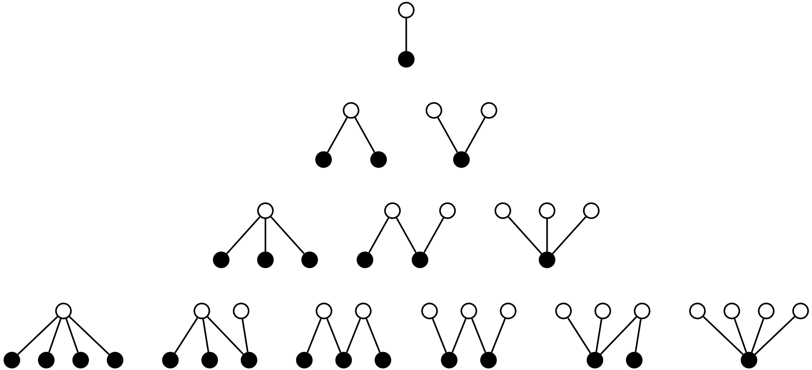


Figure: All nondegenerate maximal connected spaces with at most five elements.

I-spaces

Definition

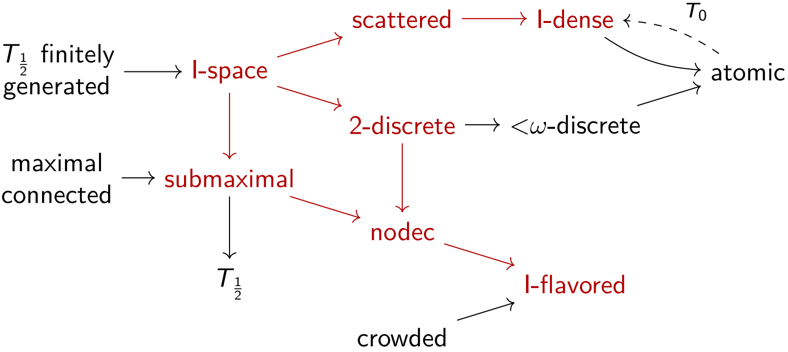
Let X be a topological space. By $I(X)$ we denote the set of all isolated points of X.

- X is an *I-space* if $X \setminus I(X)$ is discrete.
- X is *I-dense* if $\overline{I(X)} = X$.
- X is *I-flavored* if $\overline{I(X)} \setminus I(X)$ is discrete.

- I-spaces were considered in [Arhangel'skii–Collins, 1995].
- We are interested in *maximal connected I-spaces*, a class containing finitely generated maximal connected spaces.

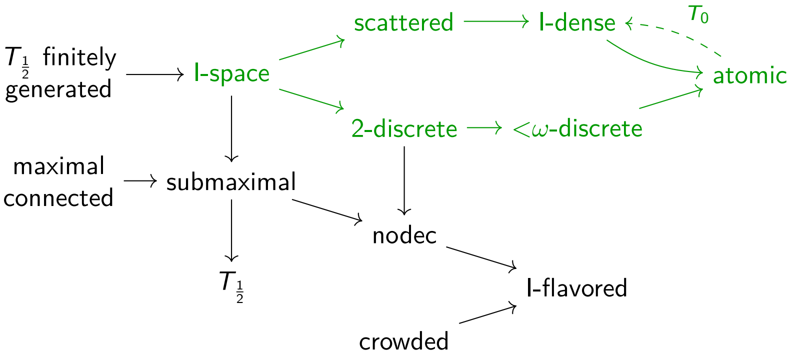
I-spaces

- We have the following implications between the classes.
- The red part is a meet semilattice with respect to conjunction.



Maximal connected I-spaces

- The green part collapses in the realm of maximal connected spaces.



Tree sums of topological spaces

Definition

Let $\langle X_i : i \in I \rangle$ be an indexed family of topological spaces, $\sim$ an equivalence on $\sum_{i \in I} X_i$, and $X := \sum_{i \in I} X_i / {\sim}$. We consider

- the canonical maps $e_i \colon X_i \to X$,
- the canonical quotient map $q \colon \sum_{i \in I} X_i \to X$,
- the *set of gluing points* $S_X := \{x \in X : |q^{-1}(x)| > 1\}$,
- the *gluing graph* G_X with vertices $I \sqcup S_X$ and edges of from $s \to_x i$ where $s \in S_X$, $i \in I$, and $x \in X_i$ such that $e_i(x) = s$.

We say that X is a *tree sum* if G_X is a tree, i.e. for every pair of distinct vertices there is a unique undirected path connecting them.

We just glue topological spaces in a way that the spaces are preserved, two spaces may be glued only at one point, and the global structure of connections forms a tree.

Tree sums of topological spaces

Proposition

A topological space X is naturally homeomorphic to a tree sum of a family of its subspaces $\langle X_i : i \in I \rangle$ if and only if the following conditions hold.

1. $\bigcup_{i \in I} X_i = X$,
2. X is inductively generated by embeddings $\{e_i \colon X_i \to X\}_{i \in I}$,
3. G is a tree, where G is the graph on $S \sqcup I$ satisfying
   - $S := \{x \in X : |\{i \in I : x \in X_i\}| \geq 2\}$,
   - $s \to i$ is an edge if and only if $s \in S$, $i \in I$, and $s \in X_i$.

Tree sums of maximal connected spaces

Definition

- We say that $A \subseteq X$ is an *I-subset* of X if it is a union of an open discrete subset and a closed discrete subset of X.
- We say that (the gluing of) a tree sum is *I-compatible* if we never glue an isolated point to a non-isolated point.

Theorem [B.]

Let X be a tree sum of nondegenerate spaces $\langle X_i : i \in I \rangle$.
The following conditions are equivalent.

1. X is maximal connected.
2. Every X_i is maximal connected and S_X is an I-subset of X.
3. Every X_i is maximal connected, $S_X \cap X_i$ is an I-subset of X_i for every $i \in I$, and the gluing is I-compatible.
4. Every X_i is maximal connected and X is essentially connected.

Tree sums of maximal connected spaces

Proposition

Let X be an I-compatible tree sum of spaces $\langle X_i : i \in I \rangle$. We have that X is $\mathcal{P}$ if and only if every X_i is $\mathcal{P}$ where $\mathcal{P}$ is

- "finitely generated",
- "an I-space",
- "finitely generated maximal connected",
- "a maximal connected I-space".

Corollary

Besides the one-point space, finitely generated maximal connected spaces are exactly I-compatible tree sums of copies of the Sierpiński space.

Extensions

There is a standard way of adding a closed disctrete set.

Definition

Let X be a topological space, Y a set disjoint with X, and $\mathcal{F} = \langle \mathcal{F}_y : y \in Y \rangle$ an indexed family of open filters on X. Let $\widehat{X}$ be the space with universe $X \cup Y$ and the following topology:

$$A \subseteq \widehat{X} \text{ is open } \iff \begin{cases} A \cap X \text{ is open in } X, \\ A \cap X \in \mathcal{F}_y \text{ for every } y \in A \cap Y. \end{cases}$$

- The space $\widehat{X}$ is called the *OF-extension* of X by $\mathcal{F}$.
- If every $\mathcal{F}_y$ is maximal, then $\widehat{X}$ is called *MOF-extension*.
- If every $\mathcal{F}_y$ contains $I(X)$, then $\widehat{X}$ is called *I-extension*.
- If both conditions hold, then $\widehat{X}$ is called *ultrafilter I-extension*.

Extensions

Remarks

- Let $X \subseteq \widehat{X}$ be topological spaces. $\widehat{X}$ is an OF-extension of X if and only if X is open dense and $\widehat{X} \setminus X$ is closed discrete nowhere dense in $\widehat{X}$.
- For I-extensions we may view the open filters $\mathcal{F}_y$ containing $I(X)$ as ordinary filters on $I(X)$. Maximal open filters containing $I(X)$ correspond to ultrafilters on $I(X)$.
- I-spaces are precisely I-extensions of discrete spaces.
- OF-extensions preserve connectedness.

OF-extensions and maximal connectedness

Proposition

Let X be a maximal connected space. For every connected $A \subseteq X$ we have that $\overline{A}$ is a MOF-extension of A.

Sketch of proof.

- Both A and $\overline{A}$ are maximal connected.
- A is open dense in $\overline{A}$ and $\overline{A} \setminus A$ is closed discrere.
- $\overline{A}$ is an OF-extension of A.
- The extending filters have to be maximal.

OF-extensions and maximal connectedness

Observation

A topological space is open-hereditarily irresolvable if and only if $\mathrm{int}(A) \cup \mathrm{int}(B)$ is dense for every its decomposition $\langle A, B \rangle$.

Proposition

An OF-extension $\langle \widehat{X}, \tau \rangle$ of a maximal connected space X by a family of filters $\langle \mathcal{F}_y : y \in Y \rangle$ is maximal connected if and only if it is a MOF-extension of X.

Sketch of proof of "$\Longleftarrow$".

- Let $A \subseteq X$ be non-open, $\tau^* := \tau \vee \{A\}$.
- WLOG $A \subseteq X$, and so $\tau^* \restriction X$ is disconnected.
- Let $\langle U, V \rangle$ be a $(\tau^* \restriction X)$-clopen decomposition of X.
- $\mathrm{int}_\tau(U) \cup \mathrm{int}_\tau(V)$ is τ-dense.
- Every maximal filter $\mathcal{F}_y$ contains exactly one of U, V.
- $\langle \overline{U}^{\tau^*}, \overline{V}^{\tau^*} \rangle$ is a τ^*-clopen decomposition of $\widehat{X}$.

OF-extensions and maximal connected I-spaces

Proposition

An OF-extension of a topological space X is an I-space if and only if it an I-extension and X is an I-space.

Corollary

Let X be a maximal connected I-space.

- An OF-extension of X is a maximal connected I-space if and only if it is an ultrafilter I-extension.
- $\overline{A}$ is an ultrafilter I-extension of A for every connected $A \subseteq X$.

Towards characterization of maximal connected I-spaces

- We have described two constructions that preserve the property of being maximal connected I-space:
  - I-compatible tree sums,
  - ultrafilter I-extensions.
- Therefore, we may build various maximal connected I-spaces inductively using the constructions.

[pictures]

- Next we shall show how to deconstruct a maximal connected I-space in order to see whether it was inductively built using the constructions.

Intersections of connected subsets

We will need the following results.

Theorem [Neumann-Lara, Wilson; 1986]

Let X be an essentially connected space. If $A, B \subseteq X$ are connected, then $A \cap B$ is connected as well.

Corollary

Let X be an maximal connected space. If $A, B \subseteq X$ are disjoint and connected, then $|\overline{A} \cap \overline{B}| \leq 1$.

Proof.
We have $\overline{A} \cap \overline{B} \subseteq (\overline{A} \setminus A) \cup (\overline{B} \setminus B)$, which is a closed discrete set since X is submaximal.

Towards characterization of maximal connected I-spaces

Definition

Let X be a maximal connected space, let $\mathcal{D}$ be a decomposition of X into connected subspaces.

- We define a graph $G_{\mathcal{D}}$ as follows: the vertices are the members of $\mathcal{D}$ and for $D \neq D' \in \mathcal{D}$ and $x \in X$, there is an edge $D \to_x D'$ if and only if $\overline{D} \cap D' \ni x$.
- We put $\mathcal{D}^+ := \{\bigcup \mathcal{C} : \mathcal{C}$ is an undirected component of $G_{\mathcal{D}}\}$.

Proposition

Given the objects above, let $D \in \mathcal{D}^+$ and let $\mathcal{C}$ be the component of $G_{\mathcal{D}}$ such that $D = \bigcup \mathcal{C}$. We have that $G \restriction \mathcal{C}$ is a tree and D is the tree sum of its subspaces $\{\overline{C} : C \in \mathcal{C}\}$ with closed discrete set of gluing points.

Towards characterization of maximal connected I-spaces

Definition

Let X be a maximal connected space. We inductively define decompositions $\mathcal{D}_\alpha$ and corresponding equivalences E_α for every α.

- $\mathcal{D}_0 := \{\{x\} : x \in X\}$,
- $\mathcal{D}_{\alpha+1} := \mathcal{D}_\alpha^+$,
- $E_\alpha := \bigcup_{\beta<\alpha} E_\beta$ for limit α.

We denote the smallest α such that $\mathcal{D}_\alpha = \mathcal{D}_{\alpha+1}$ by $\rho(X)$.

Towards characterization of maximal connected I-spaces

Theorem

In the situation above suppose that X is a maximal connected I-space. Let $D \in \mathcal{D}_\alpha$ for some α.

1. D is an I-compatible tree sum of ultrafilter I-extensions of some members of $\mathcal{D}_\beta$ if $\alpha = \beta + 1$. The ultrafilters are principal if $\beta = 0$, free otherwise.
2. D is the direct limit of $\{C \in \bigcup_{\beta<\alpha} \mathcal{D}_\alpha : C \subseteq D\}$ if α is limit.

Therefore, the members of $\mathcal{D}_{\rho(X)}$ are obtained by iteratively forming tree sums of ultrafilter I-extensions.

Proposition

Every maximal connected space having only finitely many nonisolated points is an I-space satisfying $|\mathcal{D}_1| < \omega$ and $|\mathcal{D}_2| \leq 1$. Therefore, it is a finite tree sum of free ultrafilter I-extensions of finitely generated maximal connected spaces.

Because of the previous results, a maximal connected I-space X such that $|\mathcal{D}_{\rho(X)}| \leq 1$ may be called *inductive*. We shall conclude with an example of a non-inductive maximal connected I-space.

Example

Let $f\colon X \to Y$ be a bijection between two disjoint sets, let $\mathcal{U}$ be a free ultrafilter on X. Let $\widehat{X}$ be the I-extension of X with discrete topology by the family $\langle \mathcal{F}_y : y \in Y \rangle$ where

$$\mathcal{F}_y := \{U \in \mathcal{U} : f^{-1}(y) \in U\} \text{ for every } y \in Y$$

The space $\widehat{X}$ is an example of a non-inductive maximal connected I-space.

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Thank you for your attention.

Highlights

- We are interested in *maximal connected spaces*.
- *Finitely generated* maximal connected spaces were characterized.
- Characterizing *maximal connected I-spaces* would generalize this.
- Constructions of *I-compatible tree sum* and *ultrafilter I-extension* preserve the property of being maximal connected I-space.
  We can build spaces inductively.
- Starting with points and considering how closures intersect, we obtain a *sequence of coarsening decompositions* into inductive connected subspaces.
- Not every maximal connected I-space is *inductive*.